Graphing Quadratic Functions

**Quadratic Function**
A function defined by an equation of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

**Graph of a Quadratic Function**
A parabola with these characteristics:
- $y$-intercept: $c$;
- axis of symmetry: $x = \frac{-b}{2a}$;
- $x$-coordinate of vertex: $x = \frac{-b}{2a}$.

**Example**
Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex for the graph of $f(x) = x^2 - 3x + 5$. Use this information to graph the function.

$a = 1$, $b = -3$, and $c = 5$, so the $y$-intercept is 5. The equation of the axis of symmetry is $x = \frac{-(-3)}{2(1)}$ or $\frac{3}{2}$. The $x$-coordinate of the vertex is $\frac{3}{2}$.

Next make a table of values for $x$ near $\frac{3}{2}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^2 - 3x + 5$</th>
<th>$f(x)$</th>
<th>$(x, f(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0$^2 - 3(0) + 5$</td>
<td>5</td>
<td>(0, 5)</td>
</tr>
<tr>
<td>1</td>
<td>1$^2 - 3(1) + 5$</td>
<td>3</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$(\frac{3}{2})^2 - 3(\frac{3}{2}) + 5$</td>
<td>$\frac{11}{4}$</td>
<td>$(\frac{3}{2}, \frac{11}{4})$</td>
</tr>
<tr>
<td>2</td>
<td>$2^2 - 3(2) + 5$</td>
<td>3</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>3</td>
<td>$3^2 - 3(3) + 5$</td>
<td>5</td>
<td>(3, 5)</td>
</tr>
</tbody>
</table>

**Exercises**
Complete parts a–c for each quadratic function.

**a.** Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.

**b.** Make a table of values that includes the vertex.

**c.** Use this information to graph the function.

1. $f(x) = x^2 + 6x + 8$
   - $x = -3$, $-3$
   - $x = -1$, $-1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$-4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-1</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

2. $f(x) = -x^2 - 2x + 2$
   - $x = 1$, $1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$-2$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

3. $f(x) = 2x^2 - 4x + 3$
   - $x = 0$, $0$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1$</th>
<th>$0$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>


**Graphing Quadratic Functions**

Maximum and Minimum Values  The y-coordinate of the vertex of a quadratic function is the maximum value or minimum value of the function.

<table>
<thead>
<tr>
<th>Maximum or Minimum Value of a Quadratic Function</th>
<th>The graph of $f(x) = ax^2 + bx + c$, where $a \neq 0$, opens up and has a minimum when $a &gt; 0$. The graph opens down and has a maximum when $a &lt; 0$.</th>
</tr>
</thead>
</table>

**Example**

Determine whether each function has a maximum or minimum value, and find that value. Then state the domain and range of the function.

1. $f(x) = 3x^2 - 6x + 7$
   - For this function, $a = 3$ and $b = -6$.
   - Since $a > 0$, the graph opens up, and the function has a minimum value.
   - The vertex is $x = \frac{-b}{2a} = \frac{6}{6} = 1$.
   - Evaluate the function at $x = 1$ to find the minimum value.
   - $f(1) = 3(1)^2 - 6(1) + 7 = 4$, so the minimum value of the function is 4. The domain is all real numbers. The range is all reals greater than or equal to the minimum value, that is $\{f(x) \mid f(x) \geq 4\}$.

2. $f(x) = 100 - 2x - x^2$
   - For this function, $a = -1$ and $b = -2$.
   - Since $a < 0$, the graph opens down, and the function has a maximum value.
   - The maximum value is the y-coordinate of the vertex. The x-coordinate of the vertex is $x = -\frac{b}{2a} = -\frac{-2}{2(-1)} = 1$.
   - Evaluate the function at $x = -1$ to find the maximum value.
   - $f(-1) = 100 - 2(-1) - (-1)^2 = 101$, so the maximum value of the function is 101. The domain is all real numbers. The range is all reals less than or equal to the maximum value, that is $\{f(x) \mid f(x) \leq 101\}$.

**Exercises**

Determine whether each function has a maximum or minimum value, and find that value. Then state the domain and range of the function.

1. $f(x) = 2x^2 - x + 10$
   - min., $\frac{97}{8}$; all reals;
   - $\{f(x) \mid f(x) \geq \frac{97}{8}\}$

2. $f(x) = x^2 + 4x - 7$
   - min., $-11$; all reals;
   - $\{f(x) \mid f(x) \geq -11\}$

3. $f(x) = 3x^2 - 3x + 1$
   - min., $\frac{1}{4}$; all reals;
   - $\{f(x) \mid f(x) \geq \frac{1}{4}\}$

4. $f(x) = x^2 + 5x + 2$
   - min., $-\frac{17}{4}$; all reals;
   - $\{f(x) \mid f(x) \geq -\frac{17}{4}\}$

5. $f(x) = 20 + 6x - x^2$
   - max., 29; all reals;
   - $\{f(x) \mid f(x) \leq 29\}$

6. $f(x) = 4x^2 + x + 3$
   - min., $2\frac{15}{16}$; all reals;
   - $\{f(x) \mid f(x) \geq 2\frac{15}{16}\}$

7. $f(x) = -x^2 - 4x + 10$
   - max., 14; all reals;
   - $\{f(x) \mid f(x) \leq 14\}$

8. $f(x) = x^2 - 10x + 5$
   - min., $-20$; all reals;
   - $\{f(x) \mid f(x) \geq -20\}$

9. $f(x) = -6x^2 + 12x + 21$
   - max., 27; all reals;
   - $\{f(x) \mid f(x) \leq 27\}$